On a question on graphs with rainbow connection number 2*

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Abstract

For a connected graph G, the *rainbow connection number* $rc(G)$ of a graph G was introduced by Chartrand et al. In "Chakraborty et al., Hardness and algorithms for rainbow connection, J. Combin. Optim. 21(2011), 330–347", Chakraborty et al. proved that for a graph G with diameter 2, to determine $rc(G)$ is NP-Complete, and they left 4 open questions at the end, the last one of which is the following: Suppose that we are given a graph G for which we are told that $rc(G) = 2$. Can we rainbow-color it in polynomial time with $o(n)$ colors ? In this paper, we settle down this question by showing a stronger result that for any graph G with $rc(G) = 2$, we can rainbow-color G in polynomial time by at most 5 colors.

Keywords: rainbow connection number, diameter, NP-Complete, polynomial time algorithm

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1 Introduction

All graphs considered in this paper are simple, finite and undirected. Undefined terminology and notation can be found in [2]. Let G be a graph, and $c : E(G) \rightarrow \{1, 2, \cdots, k\}, k \in N$ be an edge-coloring, where adjacent edges may be colored the same. A graph G is *rainbow connected* if for every pair of distinct vertices u and v of G, G

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has a $u - v$ path whose edges are colored with distinct colors. The minimum number of colors required to make G rainbow connected is called the *rainbow connection number* of G, denoted by $rc(G)$. These concepts were introduced in [4]. In [3] Chakraborty et al. proved that for a graph G with diameter 2, to determine $rc(G)$ is NP-Complete, and they left 4 open questions at the end, the last one of which is the following:

Suppose that we are given a graph G for which we are told that $rc(G) = 2$. Can we rainbow-color it in polynomial time with $o(n)$ colors ? For the usual coloring problem, this version has been well studied. It is known that if a graph is 3-colorable (in the usual sense), then there is a polynomial time algorithm that colors it with $\widetilde{O}(n^{3/14})$ colors [1].

Li et al. [6] and Dong and Li [5] showed that if G is a bridgeless graph with diameter 2, then $rc(G) \leq 5$, and Dong and Li [5] showed that the upper bound 5 is tight. At that time we did not realize that we could solve the above open question. Actually, from the proof of [5] we can first deduce the following result:

Lemma 1 *For any bridgeless graph G with diameter 2, we can rainbow-color G in polynomial time by at most 5 colors.*

Since a graph G with a bridge that has $rc(G) = 2$ must be composed of a bridgeless graph G' of radius 1 with a pendant edge attached at the center vertex of G', also from [5] we can get the following result:

Lemma 2 *For any graph G that is composed of a bridgeless graph G' of radius 1 with a pendant edge attached at the center vertex G', we can rainbow-color G in polynomial time by at most 4 colors.*

Since a graph G with $rc(G) = 2$ is either a bridgeless graph with diameter 2, or a graph with a bridge that is composed of a bridgeless graph G' of radius 1 with a pendant edge (the bridge) attached at the center vertex of G', as a consequence of the above two lemmas, we settle down the last question in [3] as follows:

Theorem 1 *For any graph G with $rc(G) = 2$, we can rainbow-color G in polynomial time by at most 5 colors.*

Before proceeding, we need some notation and terminology. For two subsets X and Y of V, an (X, Y)-path is a path which connects a vertex of X and a vertex of Y, and whose internal vertices belong to neither X nor Y. We use $E[X, Y]$ to denote the set of edges of G with one end in X and the other end in Y, and $e(X, Y) = |E[X, Y]|$. The k-step open neighborhood of X is defined as $N^k(X) = \{v \in V(G) | d(v, X) = k, k \geq 0\}$.

Let $N[S] = N(S) \cup S$. For a connected graph G, the eccentricity of a vertex v is $ecc(v) = \max_{x \in V(G)} d_G(v, x)$. The radius of G is $rad(G) = \min_{x \in V(G)} ecc(x)$. The diameter of G is $\max_{x \in V(G)} ecc(x)$, denoted by $diam(G)$.

2 Proof of the lemmas and theorem

Throughout this section, the input graph G is always bridgeless with n vertices, m edges, and diameter 2. We say the colorings of the following cycles containing u to be *appropriate*: let $C_3 = uv_1v_2u$ be a 3-cycle where $v_1, v_2 \in N^1(u)$, and let $c(uv_1) = 1, c(uv_2) = 2, c(v_1v_2) = 3(4)$; let $C_4 = uv_1v_2v_3u$ be a 4-cycle where $v_1, v_3 \in N^1(u), v_2 \in N^2(u)$, and let $c(uv_1) = 1, c(uv_3) = 2, c(v_1v_2) = 3, c(v_3v_2) = 4$; let $C_5 = uv_1v_2v_3v_4u$ be a 5-cycle where $v_1, v_4 \in N^1(u), v_2, v_3 \in N^2(u)$, and let $c(uv_1) = 1, c(uv_4) = 2, c(v_1v_2) = 3, c(v_3v_4) = 4, c(v_2v_3) = 5$. We know that the shortest cycles passing through u are only the above mentioned C_3, C_4 or C_5.

Based on the proof of the main result of [5], we first give an algorithm to rainbow-color a graph with diameter 2 by at most 5 colors.

Algorithm Rainbow-Color:

Step 1: Find the center vertex u of G.
Step 2: Let $B_1, B_2, \cdots, B_b \subset N^1(u)$ satisfy that for any $1 \leq i \neq j \leq b$, $B_i \cap B_j = \emptyset$, $B_i = N_{N^1(u)}[b_i]$, and $|B_i| \geq 2$. If $\bigcup_{i=1}^{b} B_i = N^1(u)$, for $1 \leq i \leq b$, let $c(ub_i) = 1$, for any $e \in E(u, B_i \setminus \{b_i\})$, let $c(e) = 2$, for any $e \in E(G[N^1(u)]))$, let $c(e) = 3$, and stop. Otherwise, proceed to the next step.
Step 3: Let $B_{b+1} \subset N^1(u) \setminus \bigcup_{i=1}^{b} B_i$ be as large as possible that satisfies that for any $w \in B_{b+1}, wb_i \notin E(G)$, but $\exists w' \in \bigcup_{i=1}^{b}(B_i \setminus \{b_i\})$ such that $ww' \in E(G)$. If $\bigcup_{i=1}^{b+1} B_i = N^1(u)$, for $1 \leq i \leq b$, let $c(ub_i) = 1$, for any $e \in E(u, B_i \setminus \{b_i\})$, let $c(e) = 2$, for any $e \in E(u, B_{b+1})$, let $c(e) = 1$, for any $e \in E(G[N^1(u)]))$, let $c(e) = 3$, and stop. Otherwise, let $B_{b+2} = N^1(u) \setminus \bigcup_{i=1}^{b+1} B_i$, proceed to the next step.
Step 4: Let $S = \bigcup_{i=1}^{b+1} B_i$.
While $B_{b+2} \neq \emptyset$,
For any any $v \in B_{b+2}$, we select a cycle R such that R is a shortest cycle containing uv, and we further choose R such that R contains as many vertices of $G \setminus S$ as possible.
If $V(R) \cap S = \{u\}$, then give R an *appropriate* coloring. Otherwise, give R an *appropriate* coloring according to the colors of colored edges of R.
Replace S by $S \cup R$.
Step 5: If $N^2(u) \subset S$, then for any $e \in E(G[N^1(u)])$, let $c(e) = 3$, and give the remaining uncolored edges by a used color and stop. Otherwise, proceed to the next step.
Step 6: Let $N^1(u) = X \cup Y$, where for any $x \in X$, $c(ux) = 1$, for any $y \in Y$, $c(uy) = 2$.

Step 7: Let all $S, T, Q \subseteq N^2(u)$ be as large as possible which satisfy that for any $s \in S, E(s, X) \neq \emptyset$ but $E(s, Y) = \emptyset$; for any $t \in T, E(t, Y) \neq \emptyset$ but $E(t, X) = \emptyset$; for any $q \in Q, E(q, X) \neq \emptyset$ and $E(q, Y) \neq \emptyset$; for any $s \in S, E(s, T \cup Q) \neq \emptyset$; for any $t \in T, E(t, S \cup Q) \neq \emptyset$. For any $e \in E(S \cup T \cup Q, X \cup Y)$, give e an *appropriate* coloring.
Step 8: If $N^2(u) = S \cup T \cup Q$, for any $e \in E(G[N^1(u)])$, let $c(e) = 3$; for any $e \in E(S, T \cup Q)$, let $c(e) = 5$, and we use a used color to color the remaining uncolored edges and stop. Otherwise, proceed to the next stop.
Step 9: Let $N^2(u) \setminus (S \cup T \cup Q) = P_1 \cup P_2$ where for any $p_1 \in P_1, e(p_1, X) = 1$, for any $p_2 \in P_2, e(p_2, X) \geq 2$.
Step 10: If $P_1 = \emptyset$, give a kind of coloring for the remaining uncolored edges with 5 colors and stop. Otherwise, $P_1 \neq \emptyset$. If $|X| \geq 2$, go to Step 12. If $|X| = 1$, go to the next step.
Step 11: Let $D_1, D_2, \cdots, D_d \subset P$ satisfy that for $1 \leq i \neq j \leq d, D_i \cap D_j = \emptyset, D_i = N_P[d_i]$. If $P = \bigcup_{i=1}^{d} d_i$, give a kind of coloring of the remaining uncolored edges of $E(G)$ with 5 colors and stop. Otherwise, let $D_{d+1} = P \setminus \bigcup_{i=1}^{d} d_i$, and give a kind of coloring of the remaining uncolored edges of $E(G)$ with 5 colors and stop.
Step 12: $\exists x \in X \setminus B_{b+2}$ such that $E(x, P_1) = \emptyset$. Let $X_1 \subseteq X$ be the set of all vertices which are adjacent to the vertices of P_1. Let $X_2 = X \setminus (X_1 \cup B_{b+2})$. Let $P_1' \subset P$ be the set of all vertices which are adjacent to the vertices of X_1, and let $P_2' = P \setminus P_1'$ and give a kind of coloring of the remaining uncolored edges of $E(G)$ with 5 colors and stop.
Step 13: For any $x \in X, E(x, P_1) \neq \emptyset$, give a kind of coloring of the remaining uncolored edges of $E(G)$ with 5 colors and stop.

Our algorithm is deduced from [5], and so we can rainbow-color the input graph by at most 5 colors, which gives the correctness of the algorithm. In the following we will examine the time complexity of the algorithm.

At first, we know that each edge is colored only once by the algorithm. Hence, the total effect of color assignments on the algorithm's running time is $O(m)$. Thus, we can get that the running time of Steps 5, 8, 10 and 13 is $O(m)$, respectively. The time complexity for finding the center vertex in any connected graph is $O(mn)$, because, it can be done by computing the eccentricity of every vertex using a Breath First Search rooted at it. For any vertex $v \in V(G)$, the running time for finding $N(v)$ is $O(n)$. Hence the running time for Steps 2, 3, 7, 9, 11, and 12 is $O(n^2)$, respectively. For any vertex $v \in N^1(u)$, the time complexity for finding $N_{N^2(u)}(v)$ is $O(n)$; for any vertex $v \in N^2(u)$, the time complexity for finding $N_{N^2(u)}(v)$ is $O(n)$; and for any vertex $v \in N^2(u)$, the time complexity for finding $N_{N^1(u)}(v)$ is $O(n)$. Hence the total running time for finding all the cycles R and coloring all the cycles in Step 4 is $O(n^4)$. In Step 6, the time complexity is only dependent on $N^1(u)$, hence the running time of Step 6 is $O(n)$. Therefore, the total running time of the our algorithm is $O(n^4)$. The proof of Lemma 1 is now complete.

We know that any graph G with a bridge that has $rc(G) = 2$ must be composed of a bridgeless graph G' of radius 1 with a pendant edge (the bridge) attached on the center vertex of G'. From the above algorithm, we know that we can rainbow-color the bridgeless graph G' with radius 1 by at most 3 colors. For the pendant edge attached on the center vertex u, we use another fresh color to color the bridge. This gives G a rainbow-coloring. Only Steps 1,2 and 3 of the algorithm are used, and the running time of them is $O(n^2)$, respectively. Hence, for any graph G that is composed of a bridgeless graph G' of radius 1 with a pendant edge (the bridge) attached on the center vertex of G', we can rainbow-color G in polynomial time by at most 4 colors. The proof of Lemma 2 is thus complete.

Finally, since a graph G with $rc(G) = 2$ is either a bridgeless graph with diameter 2, or a graph with a bridge that is composed of bridgeless graph G' of radius 1 with a pendant edge (the bridge) attached at the center vertex of G', as a consequence of Lemmas 1 and 2, the proof of Theorem 1 is complete.

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